Pi in the Sky

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Editorial Board

Contact

Pi in the Sky
PIMS University of Victoria Site Office
SSM Building Room A418b
PO Box 3060 STN CSC, 3800 Finnerty Road,
Victoria, BC V8W 3RT

250.472.4271
Fax: 250.721.8958
pims@math.uvic.ca

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Managing Editors

Anthony Quas (University of Victoria)
Email: aquas@uvic.ca

Seckin Demirbas (UBC)
Email: s.demirbas@math.ubc.ca

Production & Design

Krysten Casumpang
PIMS Communications Manager

Solutions to the 2024 Math Quickies at the end of the issue will be published in Pi in the Sky Issue 24.

WIN $100!

PIMS is sponsoring a prize of $100 CAD to the first high school student (from within the PIMS operating region: Alberta; British Columbia; Manitoba; Saskatchewan; Oregon; Washington) who submits the largest number of correct answers before June 1, 2025. Submit your answers to: **s.demirbas@math.ubc.ca**.

Welcome to Pi in the Sky

The Pacific Institute for the Mathematical Sciences (PIMS) sponsors and coordinates a wide assortment of educational activities for the K-12 level, as well as for undergraduate and graduate students and members of underrepresented groups. PIMS is dedicated to increasing public awareness of the importance of mathematics in the world around us. We want young people to see that mathematics is a subject that opens doors to more than just careers in science. Many different and exciting fields in industry are eager to recruit people who are well prepared in this subject.

PIMS believes that training the next generation of mathematical scientists and promoting diversity within mathematics cannot begin too early. We believe numeracy is an integral part of development and learning.

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Melania Alvarez, UBC
Vancouver, BC
melania@pims.math.ca

Malgorzata Dubiel, SFU
Burnaby, BC
dubiel@cs.sfu.ca

Sean Graves, U of Alberta
Edmonton, AB
sgraves@ualberta.ca

Darja Barr, U of Manitoba
Winnipeg, MB
darja.barr@umanitoba.ca

Armando Preciado Babb, U of Calgary
Calgary, AB
apprecia@ucalgary.ca

Jane Butterfield, UVic
Victoria, BC
jvbutter@uvic.ca

Patrick Maidorn, U of Regina
Regina, SK
patrick.maidorn@uregina.ca

Stavros Stavrou, U of Saskatchewan
Saskatoon, SK
sgs715@mail.usask.ca

Submission Information

For details on submitting articles for our next edition of Pi in the Sky, please visit www.pims.math.ca/resources/publications/pi-sky

Pi in the Sky is a publication of the Pacific Institute for the Mathematical Sciences (PIMS). PIMS is supported by the Natural Sciences and Engineering Research Council of Canada, the Province of Alberta, the Province of British Columbia, the Province of Saskatchewan, Simon Fraser University, the University of Calgary, the University of Lethbridge, the University of Regina, the University of Saskatchewan, the University of Victoria, the University of Manitoba, and the University of Washington.

Pi in the Sky is aimed primarily at high school students and teachers, with the main goal of providing a cultural context/landscape for mathematics. It has a natural extension to junior high school students and undergraduates, and articles may also put curriculum topics in a different perspective.

A Note on the Cover

Our cover features an artwork titled "Majolica works" by Stephen Maxwell Campbell from Manchester, England. Hand-painted using handmade oil pants, the artist uses his interpretation of perspective similar to stereographic projection.

On the work, Stephen notes:
Networks and non-euclidean geometry were the first inducement for me to plunge into Mathematics. My paintings have always come about from wondering 'what will happen if I apply this method?' or 'how can I make sense of this?' In this way I use Mathematics as a framework for approaching a subject or as a tool to solve a problem, such as "what would this look like through the back of my head?", "what if the surface of my eye was bigger than the thing I am looking at?" Being relatively new to the world of Mathematics I have to say my Mathematical tool box is rather meagre, but the more I learn the more I find myself asking "what would this look like?" *www.smcampbell.eu*

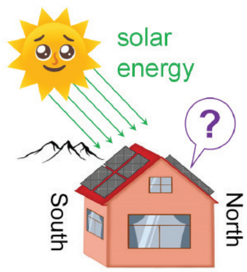
NORTH-FACING SOLAR PANELS IN CALIFORNIA

Daeho J. Lee

Westview High School

Solar Energy is very popular. We often find some buildings with solar panels installed in the North-facing roofs in the Northern Hemisphere. We want to provide a brief justification using 3D geometry.

Introduction



Using clean energy is a global movement toward sustainable cities and communities, one of the United Nations' 17 Sustainable Development Goals (United Nations (2015)). Among others, solar energy is quite popular, especially in areas with many days of sunshine, such as Southern California. In the Northern Hemisphere, many might intuitively believe that South-facing solar panels accumulate the most sunlight and, thus, in Southern California, they might expect that there are not very many North-facing solar panels. However, North-facing solar panels are easily found on Google Map. But why? Why in the Northern Hemisphere areas do they install solar panels on North-facing roofs? We wanted to provide at least one justification to answer this question with the focus on the behaviour of the solar *incidence angle*, denoted by θ, the angle between the Sun's rays the normal vector of a solar panel, using three-dimensional (3D) geometry with some trigonometric properties.

For a geographic coordinate in San Diego, California, we performed numerical examinations on $cos\theta$ accumulated over the day time (i.e., daily accumulations for the respective 365 days) and their cumulative behaviours for a one-year cycle (i.e., yearly accumulation) for West, East, South and North-facing panels, respectively, without consideration of other factors, such as weather. It is observed that the daily value of $cos\theta$ for a North-facing panel is greater than that of a South-facing panel during the 65 days around the summer solstice whereas the yearly accumulation obtained from a North-facing panel shows approximately 37.5% smaller than that from a South-facing panel.

The Model

We address several angles that are necessary to model the solar incidence angel as follows. Note that we completely rely on the well-established study results regarding solar incidence angle θ, by referring to some references (Kalogirou (2014); Duffe (2020); Kalogirou (2022)).

Declination, δ**:** the angular position of the sun at solar noon (i.e., when the sun is on the local meridian, or simply speaking, when the sun is at its highest point that day) with respect to the plane of the equator. The declination δ in degrees for any day of the year, say n, can be approximated by the following equation (refer to Kalogirou (2014)):

$$\delta = 23.45^\circ \cdot \sin\left\{\frac{360^\circ}{365}(284+n)\right\}$$

Using the simplifying assumption that each year has exactly 365 days.

Hour angle, *h*: the angular displacement of the sun east or west of the local meridian due to rotation of the earth on its axis at 15° per hour (*h*=0° at noon).

Solar altitude angle, α: the solar altitude angle is the angle between the sun's rays and a horizontal plane, which is given by the following relation (Kalogirou (2014)):

$$\sin(a) = \sin(L)\sin(\delta) + \cos(L)\cos(\delta)\cos(h)$$

where L is local latitude, defined as the angle between a line from the center of the earth to the site of interest and the equatorial plane.

Solar azimuth angle, z: the angle of the sun's rays measured in the horizontal plane from due south for the Northern hemisphere. The solar azimuth angle is expressed as

$$\sin(z) = \frac{\cos(\delta)\sin(h)}{\cos(\alpha)}$$

provided that cos(h) > tan(δ)/tan(L) (Kalogirou (2014)).

Incidence angle, θ: the angle between the sun's rays and the normal on a surface. **Fig. 1** depicts the basic angles for the angle of incidence (Kalogirou (2014)):

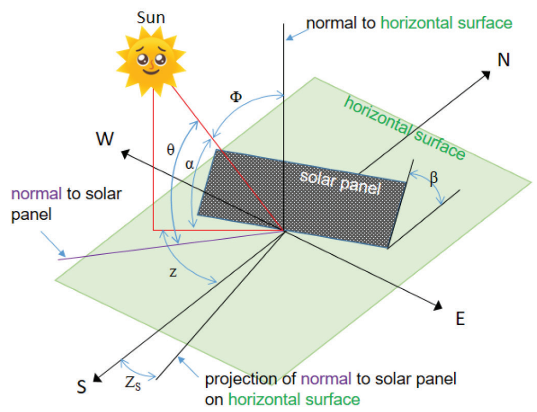


Fig. 1: Solar angle diagram for modeling the incidence angle θ for a solar panel.

where β is the surface tilt angle from the horizontal and Z_s is the surface azimuth angle which is the angle between the normal to the surface and true south.

Numerical Examples and Conclusions

We performed numerical examinations for a geographic location (32.964479°, –117.150422°) in San Diego, California. We considered β = 25° and calculated other related values accordingly. Other factors, such a seasonal weather and daily weather conditions, were not considered.

As shown in Fig. 2, the daily value of $cos\theta$ obtained from a South-facing solar panel is smaller than the other three scenarios during the 65 summer days around the summer solstice (i.e., between day 141 and day 205 where day 1 is Jan. 1st). What we found interesting is the fact that with the tilt angle β= 25° the North-facing panel can see the sun longer than the South-facing panel for many summer days, which results in the North-facing scenario's better performance.

The yearly cumulative values for the West, East, South and North-facing scenarios are approximately 0.407, 0.407, 0.478 and 0.299, respectively. This shows that the North-facing scenario performs approximately 37.5% worse than the South-facing scenario. Assuming that there are many winter days partly or fully cloudy and that $cos\theta$ is approximately proportional to the solar energy generation rate, the difference will become much smaller than 37.5%, which implies that the North-facing scenario could still be productive.

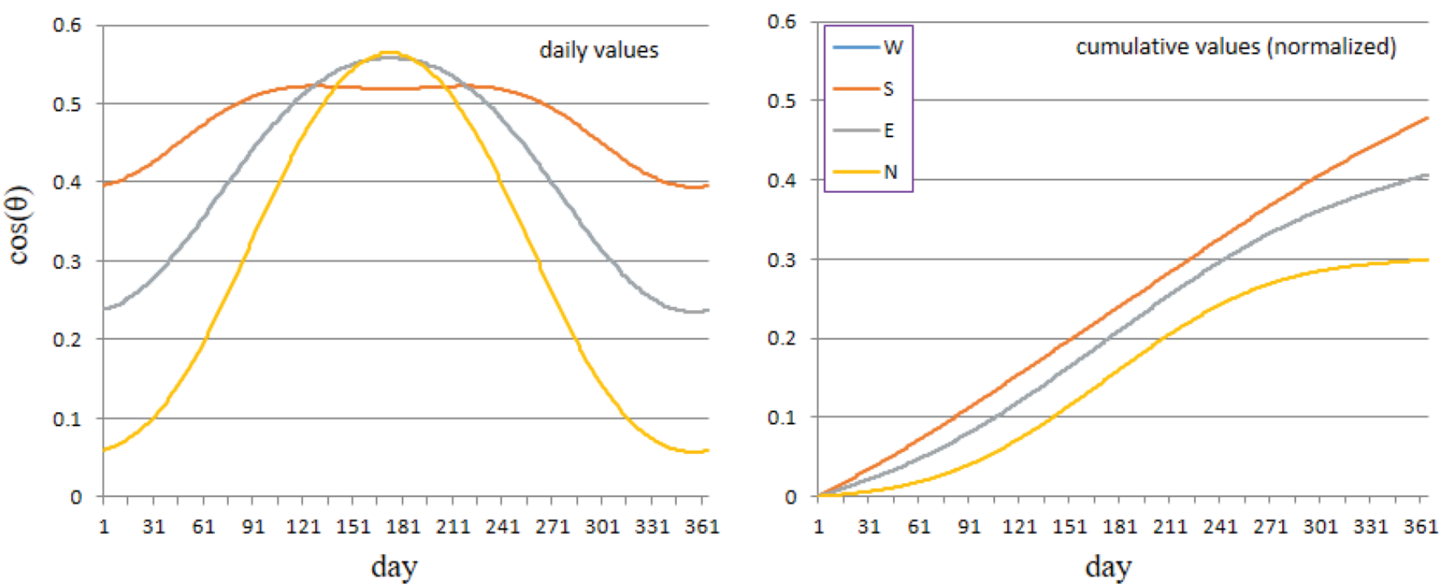


Fig. 2: Numerical examples. Daily behaviours of cos for a window of 365 days: daily and cumulative (normalized) for West, East, North and South-facing scenarios. East and West scenarios are equivalent for symmetry under our examination settings.

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SQUARING THE CIRCLE LIKE A MEDIEVAL MASTER MASON

Frédéric Beatrix*

Architect INSA Strasbourg

1 “Squaring the circle”

“Squaring the circle” is a classic math problem in geometry. It is part of the three great problems of antiquity, with the trisection of the angle and the duplication of the cube. It is the challenge of constructing a square with the same area as a given circle. The rule is the following: use only a finite number of steps, with solely a compass and straightedge (so, no ruler with measurement gradients). It basically comes down to building a squre where each side is $\sqrt{\pi}$.

In 1837, Pierre Wantzel [1] proved that only lengths which are algebraic numbers can be constructed with compass and straightedge. This is the reason why we can construct for example $\sqrt{2}$ which is the diagonal of a square 1x1, or $\sqrt{5}$ which is the diagonal of a *“Quadratum Lungum”*, a rectangle 1x2.

In 1882, Ferdinand von Lindemann [2] proved that π is not an algebraic number since it is not the root of a non-zero polynomial of finite degree with rational coefficients (we call such number “transcendental”), therefore this problem is resolved negatively: we cannot construct $\sqrt{\pi}$ using solely a compass and straightedge.

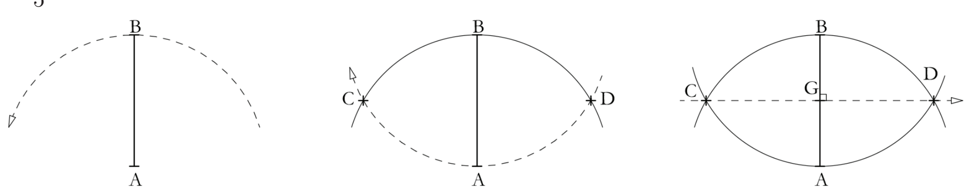
From this date, mathematicians have endeavoured to provide geometric approximation which would shine with the three following quantities: to be a simple construction, with minimum steps, to provide a good approximation of $\sqrt{\pi}$. Though so far all approximate constructions of $\sqrt{\pi}$ are borne from complex figures requiring tedious multiple steps. Mathematicians have forsaken simplicity and elegance for the sake of accuracy.

2 Definitions

Definition 1.

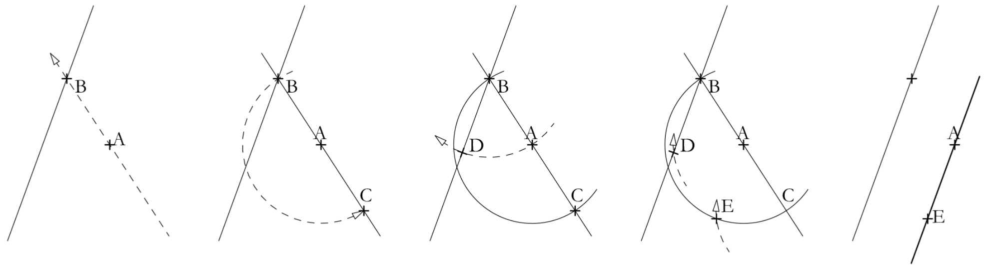
ACCURACY increases as the relative error of the constructed segment to $\sqrt{\pi}$ decreases.
ELEGANCE reflects a subjective view on the overall elegance of the whole graphic design leading to the construction of the segment which approximates $\sqrt{\pi}$.
SIMPLICITY reflects the number of simple geometric steps leading to the final length which approximates $\sqrt{\pi}$.

0 step: a given figure
1 step: draw a line or an arc
3 steps: draw a perpendicular line, or find the middle of a given segment
5 steps: draw a parallel line
11 steps: draw the $\frac{n}{5}$ fraction of a given segment

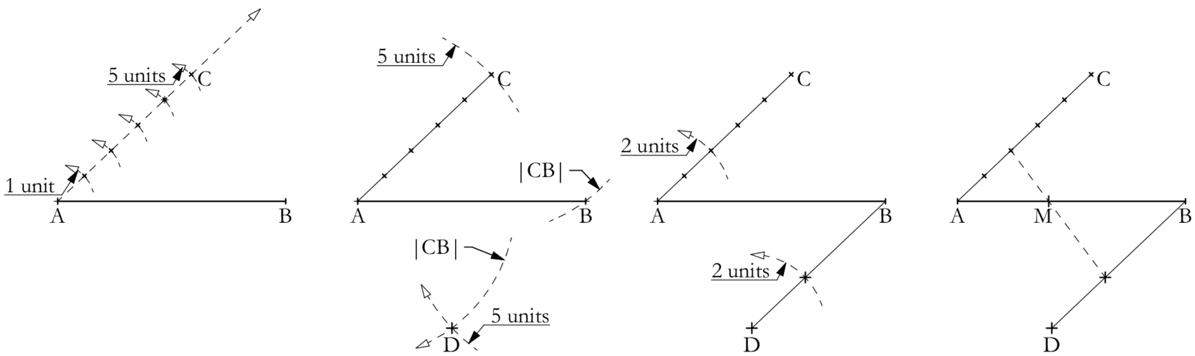


Draw a perpendicular line or find the middle of a given segment: **3 steps**

* *Frédéric Beatrix runs his own architectural firm blue.archi in Villefranche-Sur-Mer, France*



Draw a parallel line: **5 steps**



Get the n/5 fraction of a given segment: **11 steps**

Figure 1: *calculation of steps for basic geometry*

3 Accuracy vs Elegance and Simplicity

In 1913 Ramanujan Srinivasan Ramanujan [4] proposed a pretty complex figure using $\frac{355}{113}$ as an approximation of π correct to 6 decimal places. Then in 1914 he proposed yet another complicated construction, giving an approximation of π with $\sqrt[4]{9^2 + \frac{19^2}{22}}$, correct to 8 decimal places (Fig. 2). Here are the 51 construction steps:

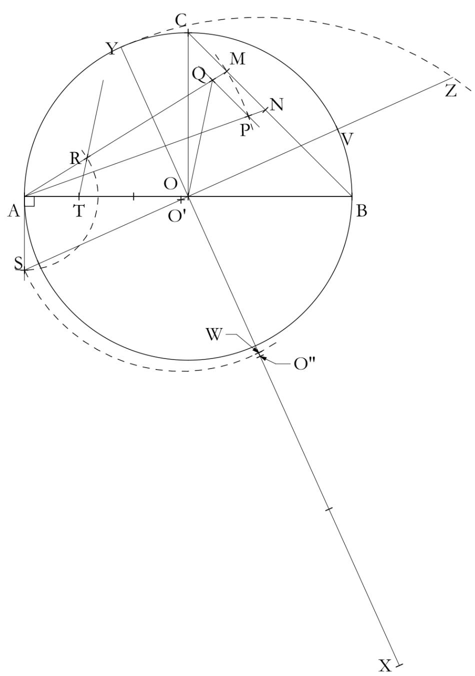


Figure 2: *Construction by Ramanujan (1914), replicated from [3].*

1: We have the given circle centred on O radius $r = 1 = |OA|$ with diameter $|AB|$
2: Bisect ACB at C (3 steps)
3: Created a 3-unit ruler from A (4 steps)
4: With the ruler trisect AO at T (5 steps)
5: Join BC (1 step)
6: Cut off from it $|CM|$ and $|MN|$ equal to $|AT|$ (2 steps)
7: Join AM and AN (2 steps)
8: Cut off AP from AN so that $|AP|$=$|AM|$ (1 step)
9: Through P draw PQ parallel to MN and meeting AM at Q (5 steps)
10: Join OQ and through T draw TR, parallel to OQ, and meeting AQ at R (6 steps)
11: Draw AS perpendicular to AO and $|AS|$= $|AR|$ and join OS (5 steps)
We have built $|OS| \approx \frac{\pi^2}{9}$
12: We now create the mean proportional between $|OS|$ and $|OB|$ by defining V, intersection of OS and the circle (1 step)
13: We identify O' middle of $|SV|$ and draw the circle radius $|O'S|$ (4 step)
14: The perpendicular to SV at O cuts this circle at W (3 steps)
15: $|OW| \approx \frac{\pi}{3}$ so we create X so that $|OX| = 3|OW| \approx \pi$ (3 steps)
16: We create Y so that $|XY| = |OX| + 1$ (1 step)
17: We build the half circle diameter $|XY|$ centred on O'' (4 STEPS)
18: The intersection of SV with this circle gives us Z (1 step) then $|OZ| \approx \sqrt{\pi}$

ACCURACY : ■■■■■ *ELEGANCE* : □□□□□ *SIMPLICITY* : ■□□□□

The latest proposal dates from 3rd August 2019: Mr. Hung Viet Chu [5] proposed a construction providing an approximation of π with $\sqrt{\frac{63}{25}(1+\frac{5}{2}\frac{15\sqrt{5}-7}{269})}$ which is correct to 9 decimal places and requires at least 68 steps (Fig. 2).

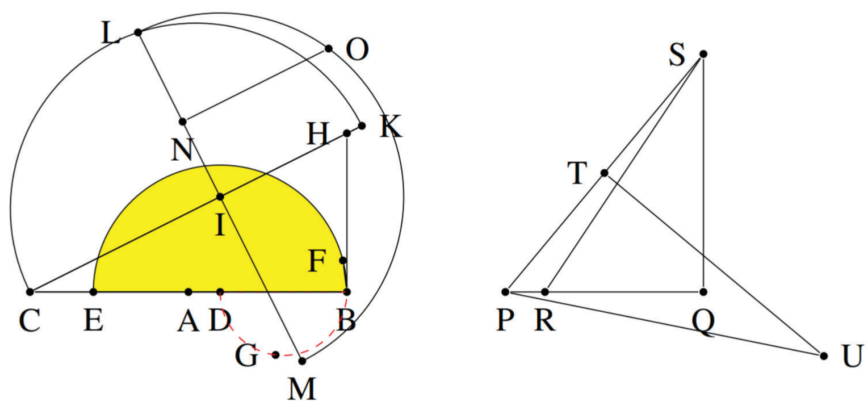


Figure 3: *Construction by Mr. Hung Viet Chu (2019) from [5]*

ACCURACY : ■■■■■ *ELEGANCE* : □□□□□ *SIMPLICITY* : □□□□□

In the same publication, Mr. Hung Viet Chu also proposed a construction base on the approximation that he attributes to Robert Dixon [6]: $\frac{6\phi^2}{5} \approx \pi$ where $\phi = \frac{\sqrt{5}+1}{2}$. The beauty of this approximation is to connect π with the "golden ratio" ϕ.

the golden ratio ϕ (phi) is naturally found in nature for example in the spirals on the pineapple, the artichoke or the pine cone (13 spirals in one direction and 8 in the other), or the spiral of seeds of the sunflower (13 in one direction and 21 in the other direction). These numbers are part of the famous Fibonacci sequence where each new term is the sum of the two previous ones: 1,1,2,3,5,8,13,21,34,55,89,144,233... And indeed, the ratio of succesive Fibonacci terms are close to the golden ratio ϕ .

The proposed graphic by Hung Viet Chu in 2019 would adequately reflect the actual complexity of the construction if it did display all the required steps and especially the geometric construction of M on AD such as $|DM| = \frac{2}{5}|AD|$. Once again, the construction of DM requires the creation of a 5-units-ruler (i.e. a line with 5 equal measurement gradients) and one could consider that it is a breach of the rule. Here are the proposed 29 steps:

1: Let $|AB| = 1$
2: Construct $BC \perp AB$ (4 steps)
3: $|BC| = 2$ (2 steps)
4: Draw the circle centered at A, radius AC, which cuts the extended AB at D (1 step)
5: Let M on AD such that $|DM| = \frac{2}{5}|AD|$ (11 steps)
6: Let N be such that $|ND| = \frac{1}{2}|DM|$ (4 steps)
7: Draw the circle taking NB to be one of its diameter (4 steps)
8: Let H be on the circle such that $H \perp NB$ (3 steps)

Then $|MH| = \sqrt{\frac{6\phi^2}{5}}$

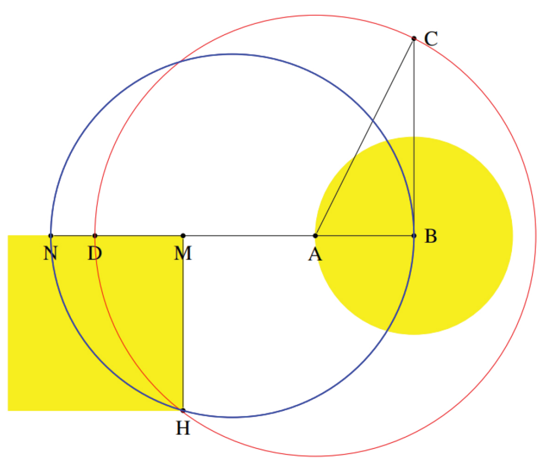
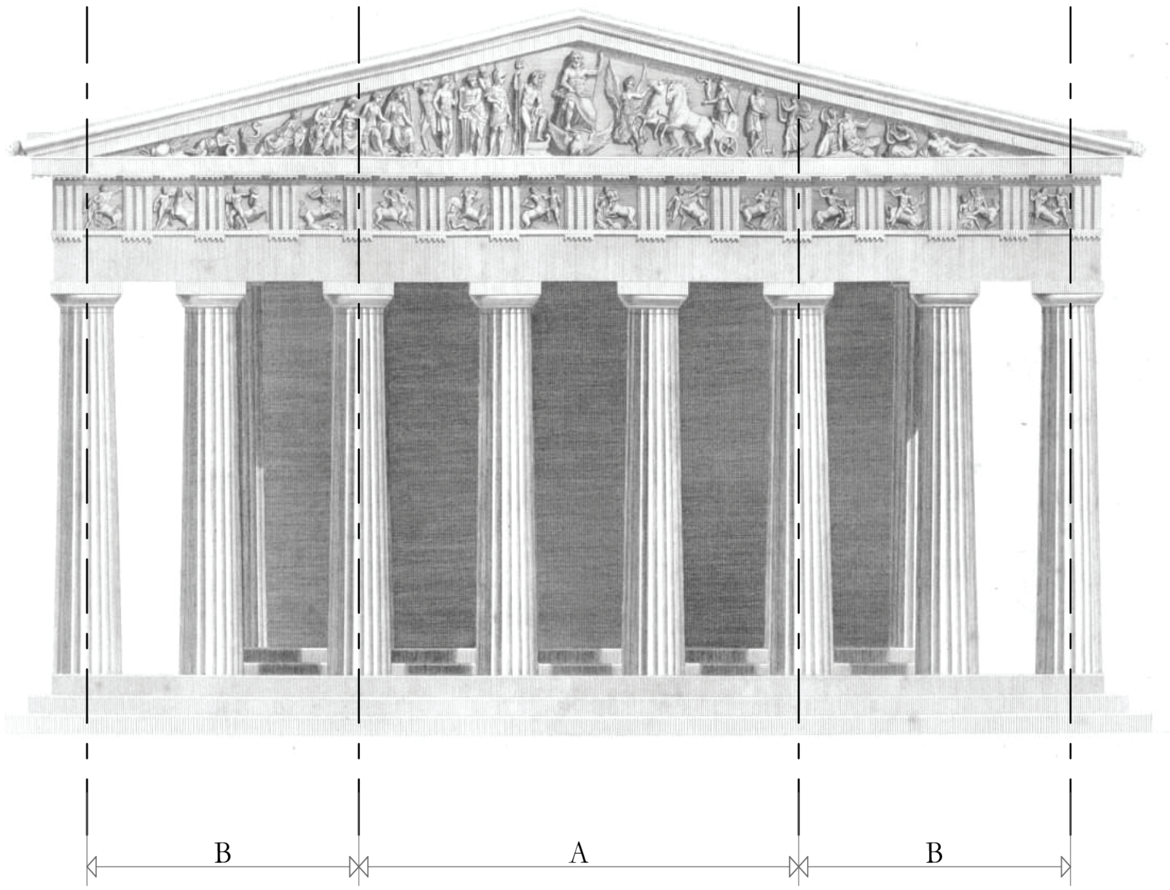


Figure 4: *Construction by Mr. Hung Viet Chu using the Dixon approximation (2019) from [5]*

$ACCURACY$: ■■■■□ $ELEGANCE$: ■■■□□ $SIMPLICITY$: ■■■□□



$$\frac{A}{B} = \varphi$$

Figure 5: *Golden Ratio ϕ and Parthenon - source for background drawing [7]*

4 Golden Ratio and Architecture

I must mention that ϕ has great relevance in architecture since antiquity as it can be measured in the Great Pyramid of Gizeh and the Parthenon for example. The apothem of the Great Pyramid at Gizeh has an angle of $\frac{14}{11}$ which is an accurate approximation of $\sqrt{\phi}$. The Parthenon is an important example since the "Golden Ratio" ϕ owes its Greek initial letter to its main sculptor and architect Phidias ($\phi\epsilon\iota\delta\iota\alpha\zeta$). Actually the figure 5 is a hint on a whole new assessment of the place of geometry in antique Greek architecture, in a coming book* we will demonstrate the underlying geometrosophy of the Parthenon which does include ϕ.

Another likely reason for the common use of ϕ by antique architects is that ϕ appears naturally in the simple construction of a Quadratum Lungum (double square) as shown in Figure 5. **For architects, the simplicity and elegance of the geometry has at least the same value as accuracy.**

*"Le Tracé Primordial: la géométrie secrete des bâtisseurs", ed. DERVY-MEDICIS, 2024

5 Proposed Dating for the Approximation $\frac{6\phi^2}{5} \approx \pi$

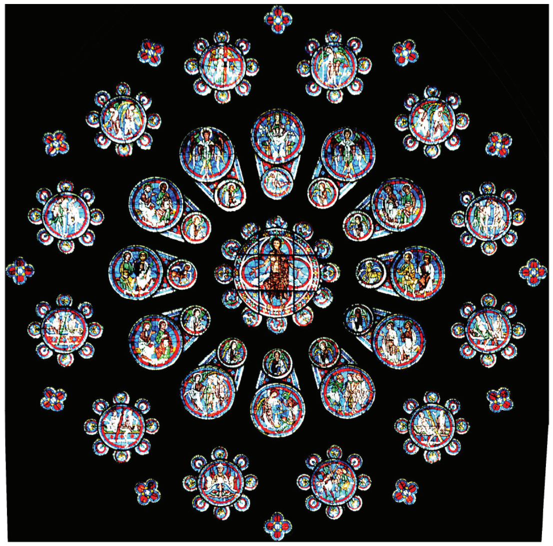


Figure 6: *Rose Window of the western façade of Chartres Cathedral, France - source: [8]*

The approximation $\frac{6\phi^2}{5} \approx \pi$ may operatively date from medieval times since it is a geometric quality of the building units used by the French builders of gothic cathedrals. At the time, the unit is the "Ligne du Roi de France" and the standard set of French medieval units are the following:

34.*lignes* = 7.64*cm* is one *Palmus minor*

55.*lignes* = 12.36*cm* is one *Palmus major*

89.*lignes* = 20*cm* = $\frac{1}{5}m$ is one *Empan* (span)

144.*lignes* = 32.36cm is one *pied* (foot)

233.*lignes* = 52.36cm = C is one French medieval *Coudée* (Cubit)

They are all numbers of the Fibonacci sequence. More specifically if you consider the *Coudée* of 233.*lignes* and the *Empan* of 89.*lignes*, then we have $C \approx \frac{\phi^2}{5}m$.

Furthermore, a Master Mason at work will necessarily find out that a French Medieval *Coudée* measures 1/6 of the perimeter of a circle diameter 5 *Empan* (that is $1m$ diameter). Indeed this property allows him to construct easily large-scale 6-petals or 12-petals "rose window" as you commonly find in the axis of the west elevation of so many cathedrals. These operative qualities translate mathematically as $\frac{\phi^2}{5} \approx \frac{\pi}{6}$ at 0.0015% which is near-perfect for operative masonry*. Comparatively the tolerance for concrete structures nowadays is around 2% in most countries.

6 Squaring the Circle Like a Medieval Master Mason

With the medieval approximation that we have presented, we propose a simple, elegant and accurate method for squaring the circle (Fig. 4). From the given circle radius 1.

1: We design *Quadratum Lungum* (double square) $ABCD$ where $|AD| = 1$ (10 steps)
2: We create the inner circle radius $\frac{1}{2}$ centred on G (1 step)
3: The intersections of the diagonals with the inner circle generates E and F (3 steps)
4: We draw the arc centred at A, radius AE, which cuts AB at H (1 step)

$FH = \sqrt{\phi^2 + (\frac{\phi}{\sqrt{5}})^2} \approx \sqrt{\pi}$ at 0.0007% that is 7mm over one kilometre. Therefore, the yellow square has nearly the same area as a circle radius 1. The level of accuracy is the same as Figure 4 by Hung Viet Chu. Though our proposal requires only 15 steps and produces a very neat and elegant figure.

* In medieval times, "operative masonry" was roughly what we now call "architecture"

- Proof -

The hypotenuse of ADC is $AC = \sqrt{AD^2 + DC^2} = \sqrt{5}$ therefore, $AH = AE = AG + GE = \frac{\sqrt{5}}{2} + \frac{1}{2} = \phi$. Since $\frac{AE}{AC} = \frac{AF}{AD}$ then $AF = \frac{\phi}{\sqrt{5}}$. Therefore, the hypotenuse of FAH is $FH = \sqrt{\phi^2 + (\frac{\phi}{\sqrt{5}})^2} \approx \sqrt{\pi}$ as desired.

$ACCURACY$: ■■■■□ $ELEGANCE$: ■■■■■ $SIMPLICITY$: ■■■■■

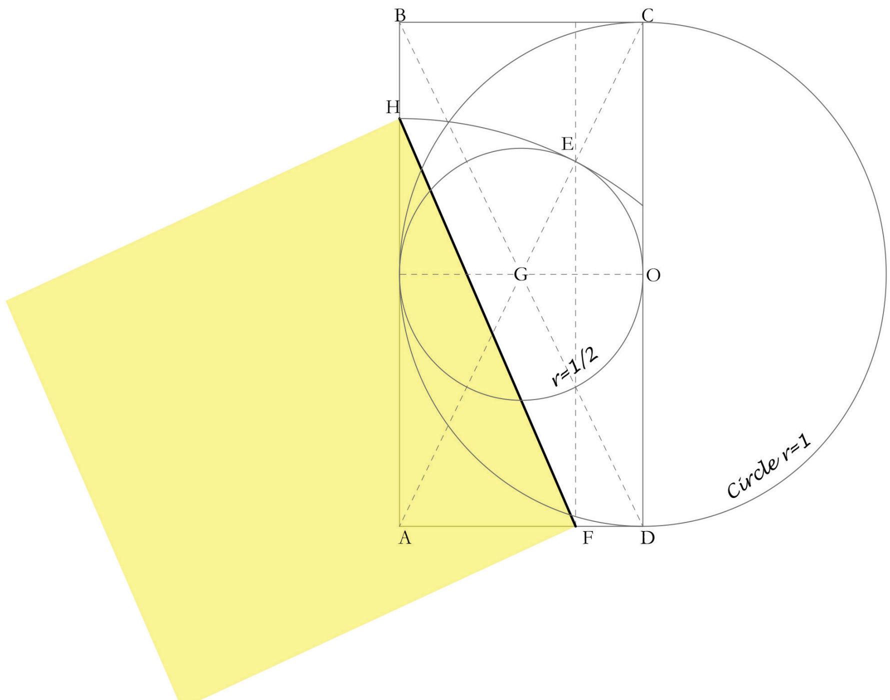


Figure 7: *Squaring the circle like a Master Mason*

7 Bonus: The Pentagram Hidden in the Quadratum Lungum

With the very same figure 5, we easily calculate that $HD = \sqrt{\phi + 2}$ and $HO = \sqrt{3 - \phi}$ (figure 6). These dimensions are respectively the isosceles side (HD) and the base (HO) of a "golden triangle" inscribed inside a circle radius 1. Therefore, you can immediately build a regular pentagon and pentagram.

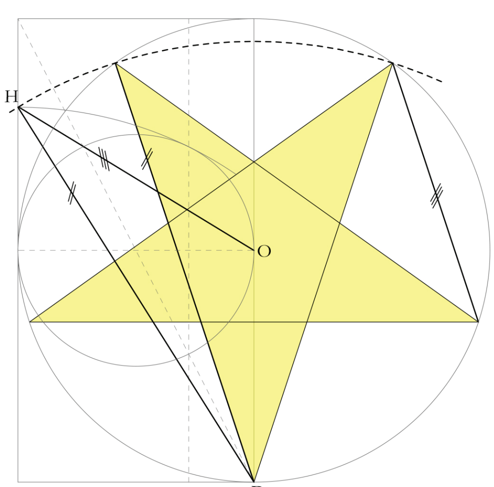


Figure 8: *The pentagram hidden in the Quadratum Lungum*

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[7] Background drawing: "Elevation of the portico restored" by Nicholas Revett (1721 - 1804)

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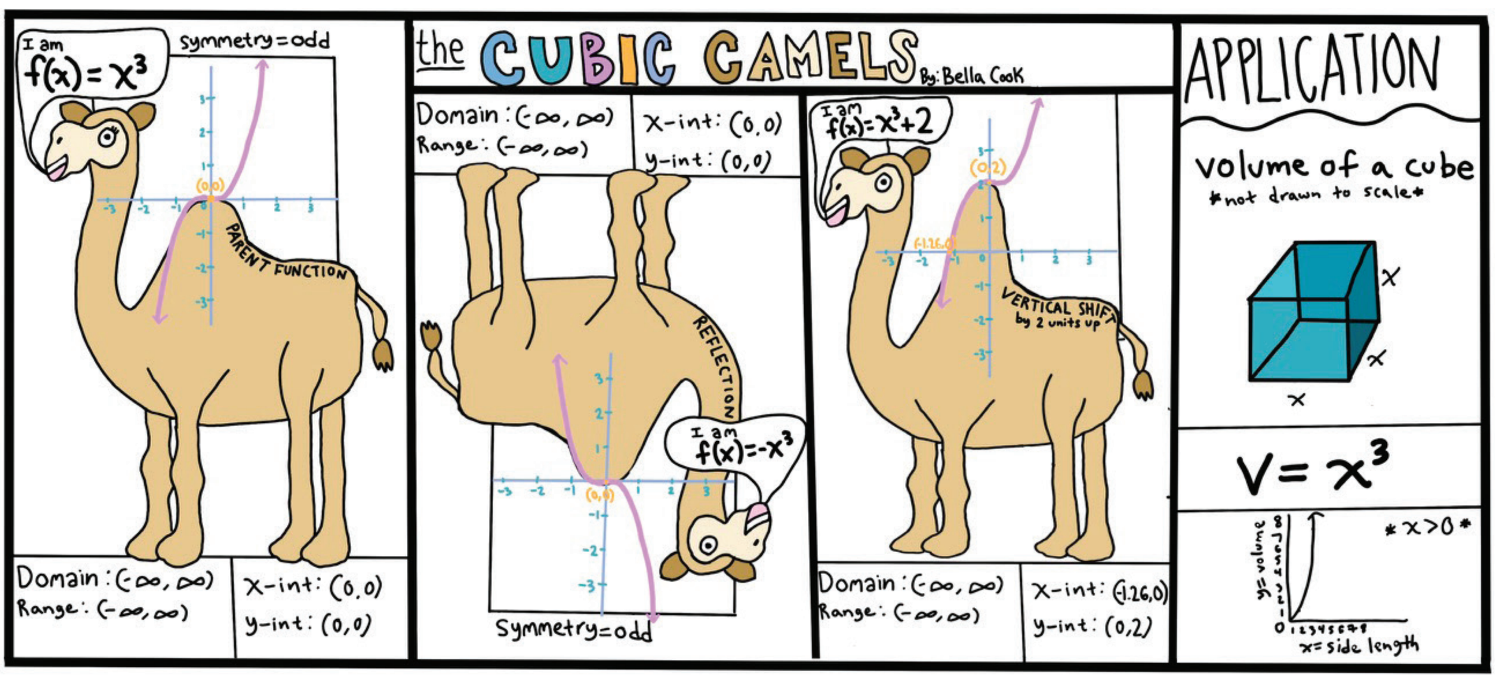


ABOUT THE ILLUSTRATOR

Bella Cook is a Junior at Detroit Country Day School and loves to combine her artistic passion with the study of mathematics.

A PERFECT MATCH: STABILITY AND THE GALE-SHAPLEY ALGORITHM

Kenneth Hou and Kimberly Hou

Kenneth Hou is a junior at Saratoga High School, California. He developed his love for cats, fencing, and mathematics in California, where he has lived his entire life.

Kimberly Hou is an undergraduate at Princeton University. She enjoys figure skating and horseback riding in her free time.

Introduction to Matchings

At first glance, man's irrational wants and needs seem like a foil to mathematics' rigorous proofs and clean definitions. However, Game Theory, an interdisciplinary field of economics and math, aims to formalize ways to meet these needs. the Prisoner's Dilemma is the most famous example of Game Theory. In this paper, we will be specifically looking at matchings and the Gale-Shapley algorithm.

In a matching problem, two different sets of agents act on their preferences to produce a matching between the two sets. Either one or both sets of agents has a set of preferences, or an ordered list in decreasing rank of the opposite set. In our example, we'll focus on colleges and applicants. The formal model of our matching problem will look like this:

- A set of n colleges, C, and a set of m applicants, A.
- Each applicant a will have a set of complete preferences over the colleges, $C_1 \succ C_2 \succ \cdots \succ C_n$. The $\succ$ notation means preference, so applicant a would prefer college C_1 to college C_2.
- Similarly, each college c will also have a set of complete preferences over the applicants.

For the above example, we will assume that there are the same number of applicants and colleges, and each college can accept a maximum of one applicant, making the example a two-sided (because both applicants and colleges have preferences over the other) one-to-one matching.

The two sided one-to-one matching problem was historically called the "marriage problem" with men proposing to women and women choosing whether or not to accept the men's proposals. However, this terminology is outdated.

Other types of matching problems include one sided matching problems and many-to-one matching problems. In one sided matching problems, only one side has preferences over the other, but the other side is indifferent. Examples of this include matching tenants to apartments. In many-to-one matching problems, one side can accept multiple members of the other side, such as how each employer will accept many employees. We will not be focusing on one-sided matchings, and our definitions will not necessarily apply to them.

Stable Matching

Now that we have defined what a matching is, can we also formalize what it means for a participant to be happy with their matchings? It's clear that not everyone will be able to get their first choice, but can we maximize everyone's satisfaction? Or is there an easier way to determine whether a matching is "good"?

Let's look at a matching that wouldn't work. Suppose Albert prefers the University of British Columbia to McGill and UBC prefers Albert to Nita. If Albert is matched to McGill and Nita is matched to UBC, then Albert and

UBC would gain more by leaving their current matches and matching with each other. It's clear that such a situation would result in an unfavourable matching.

Albert and UBC are known as a *blocking pair*. An applicant a and a college c form a blocking pair for a matching M if a prefers c to her match in M and c prefers a to their match in M. A matching is *stable* if it has no blocking pairs. Stability is a good indication of whether or not a matching system is viable long term since no pair would benefit by leaving their current match for each other.

This concept of stable matchings was formalized in 1962 by David Gale and Lloyd Shapley. In this paper, they proved that there always exists a stable matching and provided and proved their landmark algorithm for finding a stable matching. This algorithm became known as the Deferred Acceptance or Gale-Shapley algorithm. In 1984, using results from Gale and Shapley, Alvin Roth published a paper showing that the system that assigned medical school graduates to hospitals in the United States resulted in a stable matching. For their work on this problem, Shapley and Roth were awarded the 2012 Nobel Memorial Prize in Economics.

Examples of Gale-Shapley Algorithm

The steps of the Gale-Shapley Algorithm are as follows:

- Round 1: Each student proposes to his/her first-choice residency program. Each college tentatively accepts the most preferred acceptable applicant up to its quota and rejects all others.
- Round $k \geq 2$: Any student rejected at round $k-1$ applies to his/her next highest choice (if there are any). Each college considers both the new applicants and the students held at step $k-1$ and tentatively accepts the most preferred acceptable from the combined pool up to its quota, the other students are rejected.

The algorithm terminates when there are no new proposals. In fact, the algorithm terminates in finite time because there are finitely many students and finitely many colleges, and each student proposes to each of the programs at most once.

Here is how the algorithm works in real life. We have five applicants: John, Susan, Herbert, Frank, and Gabriella; three colleges: Pinewood, Saint Mary, and Good Samaritan. All the applicants and colleges have their own preferences listed in tables 1.1 and 1.2 respectively:

Table 1.1:

	John	Susan	Herbert	Frank	Gabriella
1st choice	Pinewood	Pinewood	Pinewood	Pinewood	Good Samaritan
2nd choice	Saint Mary	Good Samaritan	Saint Mary		Pinewood
3rd choice	Good Samaritan		Good Samaritan		Saint Mary

Table 1.2:

	Pinewood (1 Slot)	Saint Mary (2 Slots)	Good Samaritan (4 slots)
1st Choice	Frank	John	Herbert
2nd Choice	Herbert	Herbert	John
3rd Choice	John	Gabriella	Gabriella
4th Choice			Susan

The algorithm begins with John. First, John proposes to his first choice, Pinewood, and Pinewood has not ranked any other student higher, or any student for that matter. The algorithm tentatively matches John with Pinewood.

Next, we move onto Susan, Susan ranks Pinewood first, but Pinewood did not rank Susan. Hence, at this step, Susan is not matched with any college.

Now Herbert ranks Pinewood first, and because Herbert is ranked higher than the lowest rated accepted student, John, Herbert is matched with Pinewood, leaving John unmatched.

Frank proposes to his first choice, Pinewood. Pinewood ranked Frank as number one, higher than the current match, Herbert, who is number two. Thus, the algorithm rejects Herbert and matches Frank to Pinewood.

Gabriella ranked Good Samaritan first, and Good Samaritan ranked her too. Gabriella is now tentatively matched with Good Samaritan.

This concludes the first round. We have John, Susan, and Herbert in our unmatched list. Now the algorithm starts the second round. We consider the applicants in the unmatched list for their second choice.

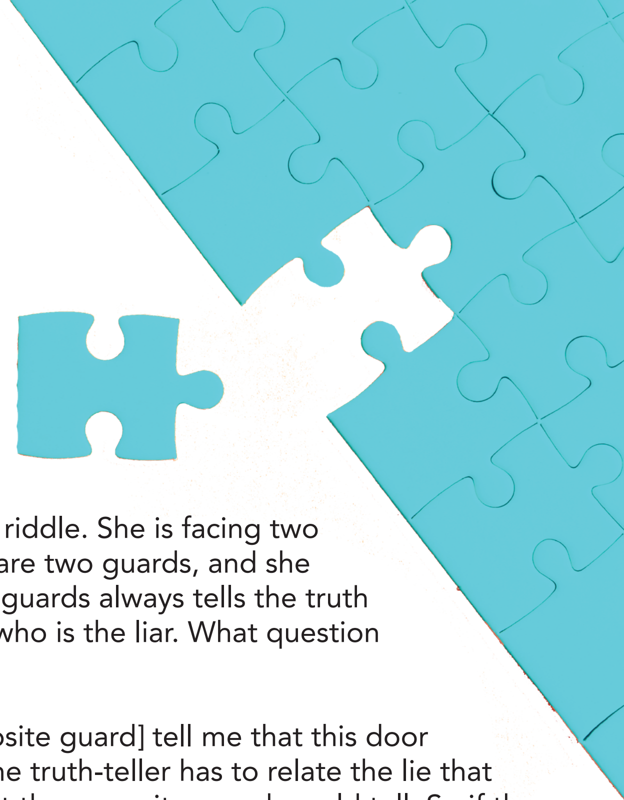
John ranked Saint Mary second and Saint Mary also ranked him. So, he is now matched with Saint Mary.

Susan proposes to her second choice, Good Samaritan. Good Samaritan also ranked her and has remaining positions, so Susan is now tentatively matched to Good Samaritan.

Herbert proposes to his second choice: Saint Mary. Saint Mary ranked John and has remaining positions so Herbert is now tentatively matched to Saint Mary.

The algorithm has stopped because every student is matched and there are no new proposals. The final matching results are listed in table 2.1:

Pinewood College	Saint Mary College	Good Samaritan College
Frank	Hubert	Susan
	John	Gabriella



THE HARDEST LOGIC PUZZLE EVER

ANTONELLA PERUCCA

A Warm-Up Logic Riddle

In the movie *Labyrinth*, the main character, Sarah, has to solve a logic riddle. She is facing two doors, one leading to a castle and one leading to certain death. There are two guards, and she can ask one yes-no question to them. Sarah also knows that one of the guards always tells the truth and the other always lies, but doesn't know who is the truth-teller and who is the liar. What question can she ask to know the door which she has to open?

Here is the question that Sarah asks: "Would he [pointing at the opposite guard] tell me that this door leads to the castle?" In both cases the answer that Sarah gets is a lie: the truth-teller has to relate the lie that the opposite guard would tell, while the liar would negate the truth that the opposite guard would tell. So if the answer is "yes" the door leads to certain death, and if the answer is "no", the door leads to the castle.

The Hardest Logic Puzzle Ever

The so-called "Hardest Logic Puzzle Ever" is the following riddle [1]:

> Three gods A, B, and C are called, in no particular order, True, False, and Random. True always speaks truly, False always speaks falsely, but whether Random speaks truly or falsely is a completely random matter. Your task is to determine the identities of A, B, and C by asking three yes-no questions; each question must be put to exactly one god. The gods understand English, but will answer all questions in their own language, in which the words for "yes" and "no" are "da" and "ja", in some order. You do not know which word means which.

Some clarifications [3]: a single god may be asked more than one question; the questions and to which god they are asked may depend on the answers to earlier questions. Random acts as either a truth-teller or a liar. You can imagine that he flips a fair coin in his head: if the coin comes down heads, he speaks truly; if tails, falsely. Finally, the gods are very intelligent beings, and can understand even very complicated logical questions in English. Moreoever, they know all their identities, plus common knowledge such as 1 + 1 = 2.

Notice that the riddle is very complicated mostly because of the presence of the Random god (whose answers convey no information), and because of the language barrier.

The Riddle Without the Language Barrier

For the moment, we suppose that we can understand the gods's language, so that - by translating - the answers to our questions are either yes or no. We start with some preliminary remarks.

- If you know that a god is either True or False (in other words, which is not Random), how can you determine his identity? This is very easy, you ask him a question for which you know the answer already, and you check if he tells the truth or not. One possible question is "Is 1 + 1 = 2?". True would answer "yes" and False would answer "no".

- If you know that a god is True, you can determine the identity of the other two gods by asking him (pointing at one of the other two gods), "Is this god Random?" If the answer is "yes", the god is indeed Random and the remaining god is False. If the answer is "no", the god is False and the remaining god is Random.

- If you know that a god is False, you can determine the identity of the other two gods by asking him (pointing at one of the other two gods) "Is this god Random?" If the answer is "no", the god is indeed Random and the remaining god is True. If the answer is "yes", the god is True and the remaining god is Random.

By the above, it suffices to ask one first question to find a god which is not Random. Indeed, we can address that god our second question "Is 1 + 1 = 2?" to determine whether he is True or False. Then we inquire again by him whether "Is this god Random?", by pointing at one of the other two gods.

The question to find a god which is not Random is slightly more complicated. We can ask any of the gods, pointing towards one of the two other gods: "Are you True if and only if this god is Random?" The "if and only if" is a logical way to put together two assertions: the global assertion is true either when both assertions are true or when both assertions are false. Let us analyze the possible answers. According to the answers, we are going to determine one god which is not Random.

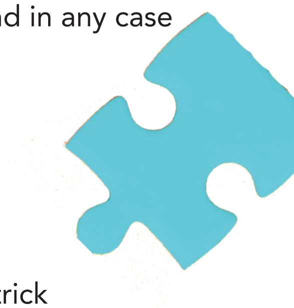
- If we are asking True, then an affirmative answer confirms that we are pointing at Random, while a negative answer means that we are pointing at False.
- If we are asking False (which lies), then the affirmative answer confirms that we are pointing at Random, while a negative answer means that we are pointing at True.
- If we are asking Random, then the answer does not carry any information. However, the god we are pointing at is not Random, and the same holds for the third god (neither the one we are interrogating, neither the one we are pointing at).

In any case, if we receive an affirmative answer, then the third god (neither the one we are interrogating, neither the one we are pointing at) is surely not Random. If we receive a negative answer, then the god we are pointing at is surely not Random.

The General Riddle

Now we consider the true riddle in which the gods answer "da" and "ja" and we have no clue what that means. We keep the same strategy as above, by varying the questions a bit. Namely, we start each question by adding "The word 'da' means 'yes' if and only if...". The dots stand for the questions as above, and in any case the dots stand for an assertion.

- We get an answer "da" to the modified question if the god would have replied "yes" in English to the original question.
- We get an answer "ja" to the modified question if the god would have replied "no" in English to the original question.



These modified questions then allow to know the English answer to the original questions. This trick allows us to solve the general riddle. Notice that this solution is basically the one given in [2].

For variations of the riddle, we direct the reader to the English Wikipedia page [3].

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Photo: @sorapop

THE CLOCK GAME:
PHRASING A VALID EQUALITY FROM AN ARBITRARY SEQUENCE OF DIGITS

Samuel Baltz* | March 1, 2023

THE CLOCK GAME

Many bored students who have watched the minutes tick by on a classroom clock will recognize the following game. Consider any sequence S of digits, together with a set O of allowable binary operations. Using the allowable operations any number of times, placing only one equals sign between any two digits, and always separating the digits with either an operation or an equals sign, the challenge is to create as many true statements as possible. Under appropriate constraints, S represents the sequence on the face of a digital clock, which every new minute provides a fresh opportunity to kill time.

For example, suppose we have allowed ourselves the set $O = \{+,-\}$, and somehow obtained S=123. Ideally, this sequence should appear in the wild. It might be the page you were on when you put down your book, or a Toronto commuter might read it off the Sherway bus at Kipling Station. Then the set of all possible attempts at equalities is as follows:

EQUALITY	TRUTH VALUE
1 = 2 + 3	0
1 = 2 – 3	0
1 + 2 = 3	1
1 – 2 = 3	0

* *Massachusetts Institute of Technology, sbaltz@umich.edu.* *I am very grateful to Rutger Campbell for detailed feedback that made this piece much better, to Neil Warnock for independently inventing this game and playing it with me when we were kids, and to Elaine Koppelman Eugster and Michael Thompson-Brusstar for very helpful conversations.*

To make sure we are clear on the rules of the game, consider an example of what is not allowed. Suppose we find the sequence S = 1110, and we offer the attempted solution 1 = 1 = 01. This idea breaks three important rules:

1. We are only permitted to use one equals sign
2. We cannot reorder the ditigs of the sequence
3. We cannot combine digits to create a new number

If the set O of allowed operations can be used to translate some sequence S into a valid equality, then we might call S *phraseable* under O. If O cannot be used to translate S into a valid equality, then S is not phraseable under O.

Which sequences are phraseable, and what are the characteristics of phraseable sequences? To answer these questions, I show results from brute force solutions of all sequences of length 4 that have integer digits from 0 to 9. We might very often encounter sequences that can appear on 24 hour clocks. Let's start by considering the four traditional binary operations of arithmetic: addition (+), subtraction (–), multiplication (*), and division (÷), and I first show results from the simplified game in which these operations can only be applied to the number sequence in order from left-to-right. I then also consider the arithmetical operators with any order of operations, as well as exponentiation (^) in addition to the operations of arithmetic.

WHAT SEQUENCES ARE PHRASEABLE?

Let's first pretend that we found a sequence S on the face of a 24 hour clock. So, consider a sequence S of length 4, where the first pair of digits represents an hour and the second pair of digits represents a minute; S can appear on a 24 hour clockface if the first pair of digits does not exceed 23, and the second pair of digits does not exceed 59. The natural question is: which times can be translated into valid equalities under the default set of arithmetical operators? Placing the equals sign between any two digits, and applying any combination of the four standard arithmetical operations (+, −, *, ÷) from left to right, is it possible to translate a given time into a valid equality in at least one way?

Figure 1 shows which times are phraseable on a standard 24 hour clockface. To make the figure, I wrote a Python program that checks every combination of hours and minutes to see whether or not some solution exists, and saves the results in a SQL database. Then, I read them into the programming language R. There I generated a matrix that has a 1 in every index that corresponds to a phraseable time, and a 0 otherwise. Finally, I replaced every 1 with the hexadecimal code for a dark colour, and 0 with a light colour. Then I used the rect() command in R to add a rectangle to the plot for every hour and minute, and coloured them according to the corresponding colour value in the matrix.

photo: @iseeghoststoo

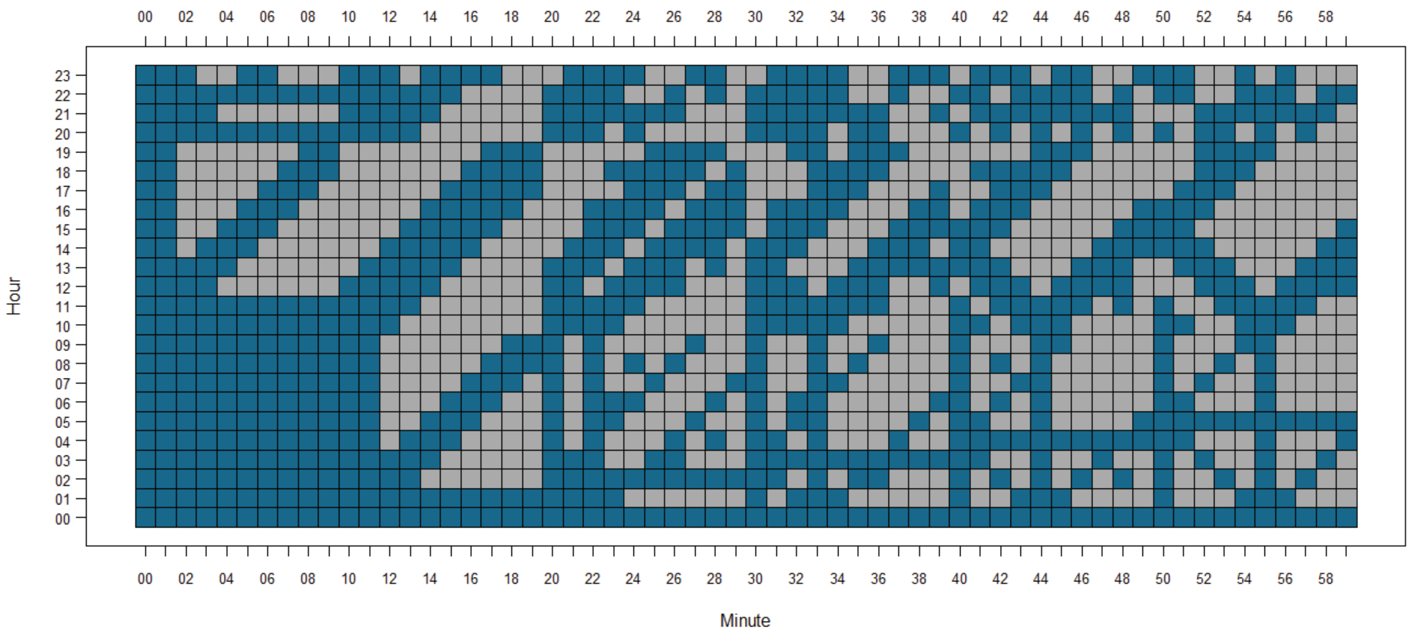


Figure 1: *Phraseable times, operations evaluated Left-to-Right, using $O = \{+, -, *, \div\}$. Each cell is a combination of an hour and a minute on a 24 hour clock. Times which are phraseable (at least one valid equality can be constructed) are blue, and times which are not phraseable are grey. The results are obtained by applying any combination of the operators {+, –, *, ÷} to a sequence and placing an equals sign anywhere, and evaluating the operations from left to right.*

The local regularities in Figure 1 combined with its chaotic overall impresions make it reminiscent of certain cellular automata [1], or some almost-regular images of Mandelbrot [2]. However, an enterprising player – say, someone who finishes their dinner and retires to the living room at 19:00 to play our game, only to discover that most of the time they cannot make any valid equalities – might wish to find a way to make Figure 1 less sparse. This inspires two complications: considering the order of operations, and expanding the set of allowed operations.

Permitting operations to be applied in an order other than left to right gives a very natural avenue for increasing the possibility of success in the game. To see why this is appropriate, notice first that the impossibility of some times in this figure appears quite natural, while the impossibility of other times is more surprising. For example, it is likely not surprising that the sequence S = 1911 does not give rise to any valid expression using only arithmetical operators. But what about the sequence S = 1222? By the strict left-to-right order of operations, there are 48 ways of attempting to form a valid equality from this sequence, and every one of them is false. It would be natural to try to form the statement 1 + 2 ÷ 2 = 2, but this is not valid if we are restricted to applying operations from left to right: 1+ 2 = 3, and $\frac{3}{2} \neq 2$. Instead, we wish to compute 1+ (2÷2) = 2. So, there are good reasons to relax the requirement that operations are applied in order from left to right. To similarly motivate the inclusion of exponentiation, notice two visible patterns in which sequences are not phraseable under the basic left-to-right arithmetical setup. First, numbers with only one zero (mostly those in the bottom-right of Figure 1) are rarely phraseable. Second, numbers which include a mix of small digits and large digits are also rarely phraseable. One operation can conveniently bridge both of these gaps: the binary operation of exponentiation will allow zeroes to be used more often in constructing valid equalities, and can frequently connect small numbers to big numbers in order to make the game more interesting.

Figure 2 shows the phraseable times when any order of operations is permitted.

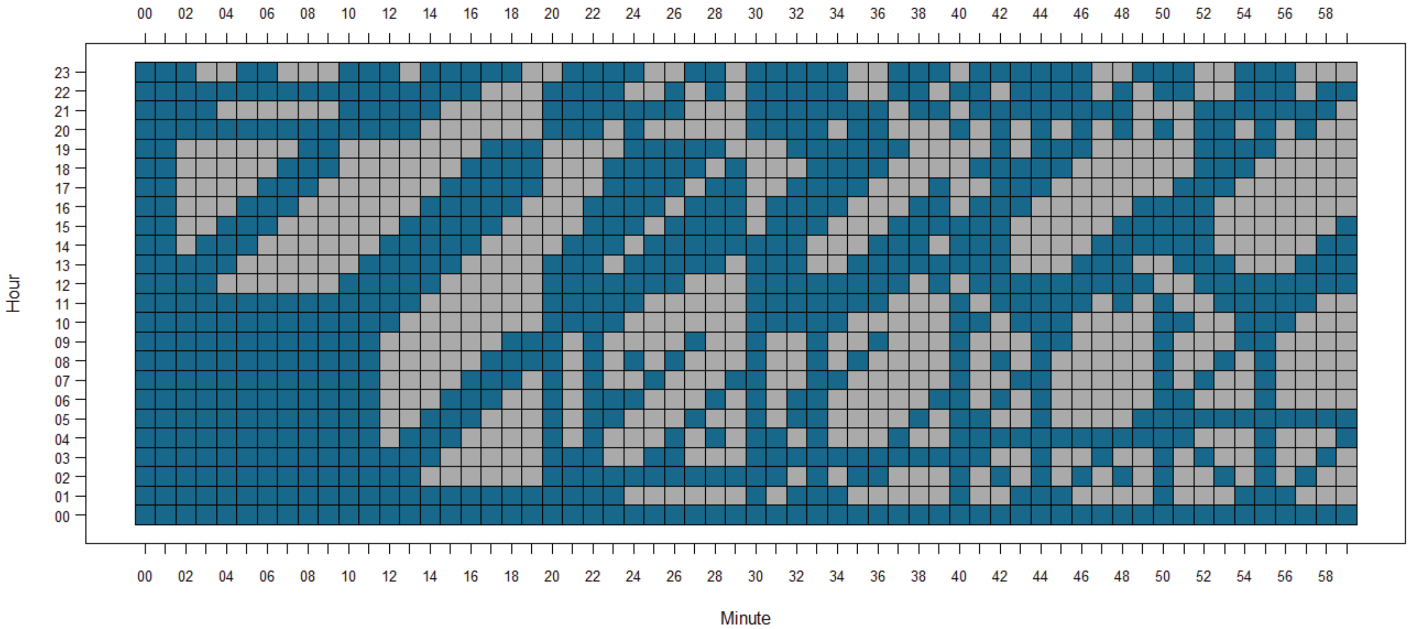


Figure 2: *Phraseable times on a 24 hour clockface, operations evaluated in any order, using $O = \{+, -, *, \div\}$.*

Figure 2 only very slightly increases the number of phraseable times (but note that differences do exist; the motivating example, S = 1222, is now phraseable). More successful is Figure 3, which shows phraseable times with a left-to-right order of operations and the expanded operation set {+, –, ⋆, ÷, ^}. If you're curious about why not use exponentiation with any order of operations, consider the difference between explicitly computing 2^3^5^9 from left to right and computing it with any order of operations.

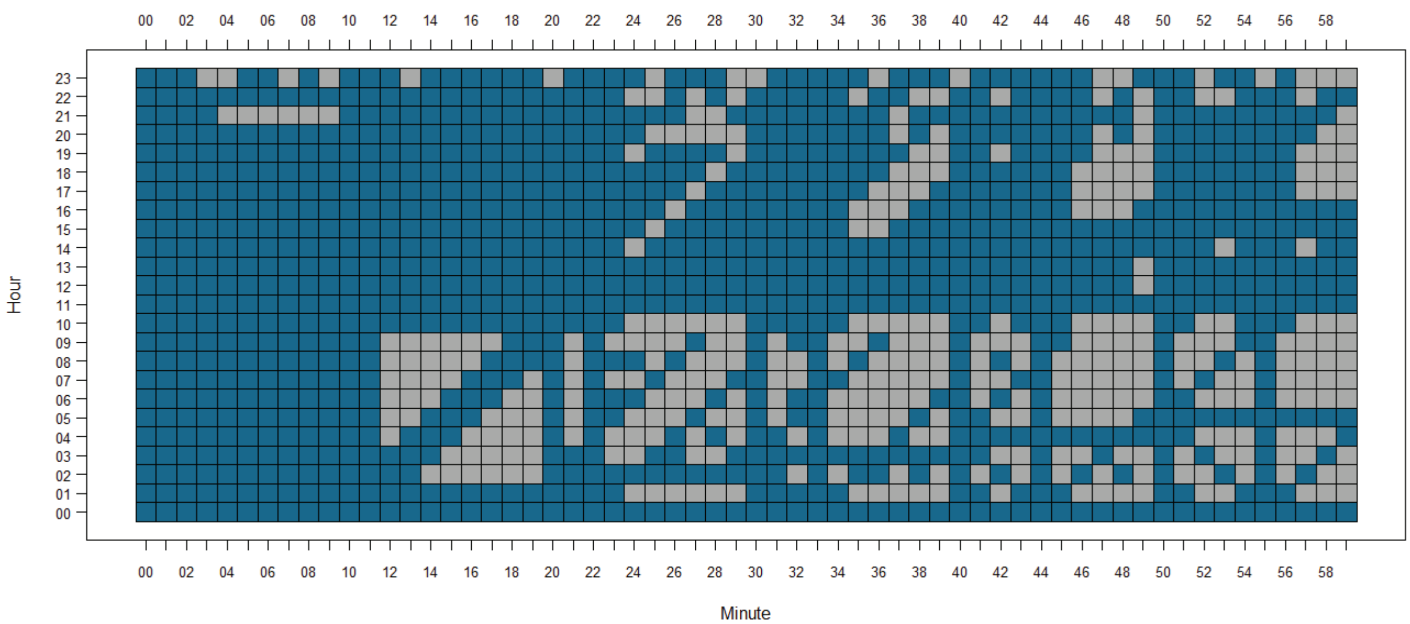
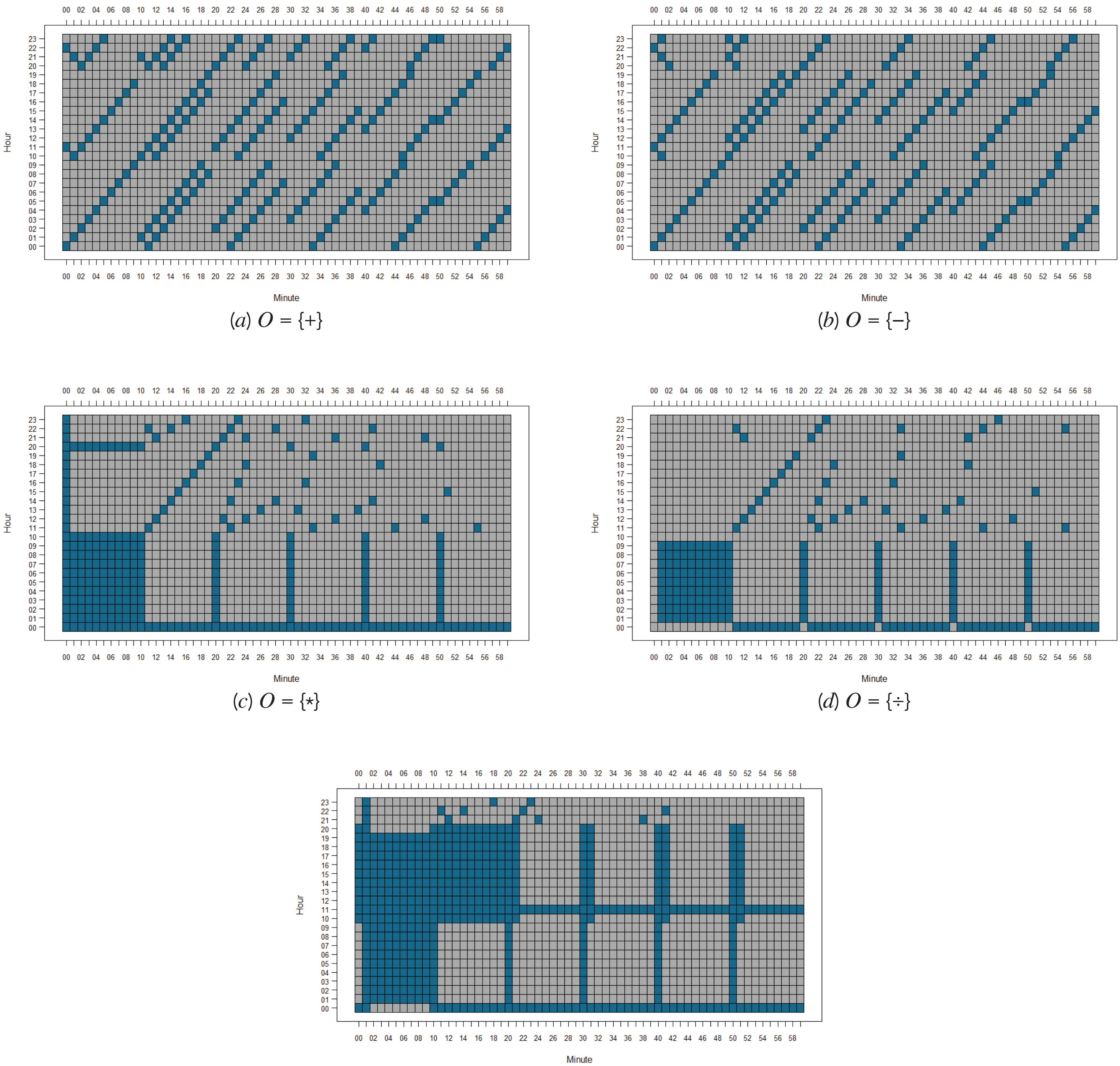


Figure 3: *Phraseable times on a 24 hour clockface, operations evaluated Left-to-Right, using $O = \{+, -, *, \div, \hat{}\}$. Each cell is a combination of an hour and a minute on a 24 hour clock. Times which are phraseable (at least one valid equality can be constructed) are blue, and times which are not phraseable are grey. The results are obtained by applying any combination of the operators {+, –, ⋆, ÷, ^} to a sequence and placing an equals sign anywhere, and evaluating the operations from left to right.*

So far, all of these figures have used the full set of permitted operations. But a player might wonder which operations are most likely to yield many phraseable sequences. Figure 4 splits the image into the usefulness of each permitted operation individually, so it considers the cases where the operation set is, respectively, $O = \{+\}$, $O = \{-\}$, $O = \{*\}$, $O = \{\div\}$, and $O = \{\wedge\}$.



(*a*) $O = \{+\}$

(*b*) $O = \{-\}$

(*c*) $O = \{*\}$

(*d*) $O = \{\div\}$

(*e*) $O = \{\wedge\}$

Figure 4: *Phraseable times on a 24 hour clockface, operations evaluated Left-to-Right, by each operation. Each cell is a combination of an hour and a minute on a 24 hour clock. Times which are phraseable (at least one valid equality can be constructed) are blue, and times which are not phraseable are grey. The results are obtained by applying only one of the operators {+, −, ⋆, ÷,^} to a sequence and placing an equals sign anywhere, and evaluating the operations from left to right.*

Note that the preceding images are more than the sum of their parts: times can be phraseable under a large operation set like O = {+, −, ⋆, ÷,^ } which are not phraseable under any one of the sets O = {+}, O = {−}, O = {⋆}, O = {÷}, or O = {^}. One such example is S = 2258, which requires a combination of these operations in order to be phraseable: The unique valid equality for S = 2258 under O = {+,−,⋆, ÷,^} is 2 = 2 ⋆ 5 − 8, and the operations must be evaluated from left to right. However, any time which is phraseable under one operation must be phraseable under any set O which includes that operation, since we could simply pick that operation repeatedly from O.

In Figure 4, we begin to see how the chaotic patterns in the previous figures have arisen, although clearly the combination of all operations is far more than the composite of the subfigures in Figure 4. Most notably, many sequences of large numbers (in the top-right of Figure 3) become phraseable only when a combination of operators are permitted. One notable high-level pattern is that, with few exceptions, the operations + and − enable long diagonal stripes of phraseable sequences where the differences or sums of numbers can be made equal, whereas the operations ⋆, ÷, and ^ provide either vertical or horizontal stripes of solutions, or large contiguous blocks of them.

FURTHER IDEAS

- What are the best binary operations to add? Is there a common binary operation not included here that makes all sequences containing a particular combination of numbers phraseable?
- Can you modify the game to include other types of operations? For example, what happens if I am allowed to take the limit of a sequence in which I apply a unary operation to one digit n times, as $n \to \infty$?
- What if instead of insisting on exact equalities, we take a sequence, place an inequality between any two digits, and attempt to obtain the smallest difference possible? Or, try to obtain a disastrously immense difference?
- This problem is similar to Crazy Sequential Representations [3, 4]. Take a look at those papers. How are Crazy Sequential Representations different from the clock game?
- If you liked this game, you'll like the calculations that Rachel Riley does on the television program *Countdown*.

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Photo: @jankaoh

Smarties® Sandwiches

Lukas Beyerlein, Dot Crumlish, Rezza Hadian, Albert Lu, Luca Nijim, Maximilian Niebur, Aiden Novick, A. Gwinn Royal, Amanda Serenevy, Samvar Harshil Shah, Shlomo Sloman, Jasmine Zhang, Stephen Zhang

Introduction

The problem below is quoted from an email from Shoshana Sloman, sent on October 26, 2020.

It has been observed that packs of Smarties® candies can always be evenly divided into five "sandwiches", where a sandwich consists of two same-coloured Smarties® with one different-coloured candy for the filling. When I say "always", I mean that people who have been playing this "game" for many years have never encountered a roll which couldn't be so divided.

I did some research and discovered that there are six possible colours/flavours of Smarties® (orange, green, red, purple, white, and yellow), and that there are 15 per roll. Assuming they were evenly distributed, it makes sense that it would always be possible to create five sandwiches.

But further research revealed that the six different colours are mixed up in one huge vat before being randomly placed into individual rolls. So, practically speaking, they end up fairly evenly distributed, but it's theoretically possible to end up with combinations that could ***NOT*** *be divided into sandwiches, such as 15 of one colour.*

My thought was that the chances of ending up with an unsandwichable roll would be vanishingly small, and that is why people never observe it. In order to figure this out, I wanted to know the number of sandwichable versus unsandwichable combinations.

Can you tell me how this problem should be approached?

Terminology and Examples

As mentioned in the e-mail above, a Smarties® roll comprises 15 candies randomly selected from the following six colours: red (R), orange (O), yellow (Y), green (G), purple (P), and white (W). Incidentally, the red tablet appears pink and is sometimes described as such; however, we will refer to it as red (R) to avoid confusion with purple (P). A roll can be classified as sandwichable or unsandwichable, depending on whether it can form five sandwiches. A sandwich is a group of three candies consisting of a pair of candies of the same colour called a bun and one filling of a differing color from the color of the bun.

A roll with six green, five yellow, one red, and three purple Smarties® is an example of a sandwichable roll. We could arrange the candies into five triplets, each a sandwich. For example, the arrangement GRG, YPY, GYG, PYP, GYG shows that this roll is sandwichable. A roll with three yellow candies and the rest orange cannot be made into five sandwiches, as there are not five non-orange candies to be the fillings. This is an example of an unsandwichable Smarties® roll. As we will show later (Corollary 1), having more than ten candies of a single colour is the necessary and sufficient condition for unsandwichability. We will use this condition to count unsandwichable rolls in the following sections.

Ordered Versus Unordered Rolls

There are two ways to count the total number of possible Smarties® rolls. At first, it might seem reasonable to count rolls without regard to order since we disassemble the rolls to form sandwiches anyway. In other words, we might count PPPPPPPPPPPPPPW and WPPPPPPPPPPPPPP as the same case given their same colour profile.

However, it turns out that this approach is incorrect. We actually need to count ordered rolls to find the correct probabilities. This is because while all ordered rolls are equally likely to occur, not every color profile is equally likely. To make this clear, suppose there were only two colours, Green and Purple, and that there were only 4 candies in the roll. There would then be $2^4 = 16$ different possible ordered rolls, but there are 5 possible rolls if we count each colour profile once. One of the ordered rolls has 4 Green tablets (GGGG), 4 of them have 3 Green and 1 Purple (GGGP, GGPG, GPGG, PGGG), 6 of them have 2 Green and 2 Purple (GGPP, GPGP, GPPG, PGGP, PGPG, PPGG), 4 of them have 1 Green and 3 Purple (GPPP, PGPP, PPGP, PPPG), and one of them has 4 Purple (PPPP). Because colour profiles are not equally likely, we must count ordered Smarties® rolls to obtain the correct probability that a given colour combination is manufactured.

Counting Unsandwichable Rolls

We will prove below that, if a Smarties® roll contains 15 candies in at most 6 distinct colours, it is unsandwichable if and only if there are more than 10 candies of any one colour. To count the number of rolls that have k candies of one colour, we can first count the number of ways to place those k candies into the ordered slots of the roll, then multiply by 6 for the colour options for the dominant colour, then count the ways we can fill the remaining $15 - k$ slots in the roll. This gives us the following expression for each case:

$$\binom{15}{k} \cdot 6 \cdot 5^{15-k}$$

This approach will work as long as $k \geq 8$. Below that value, this expression would over count cases where there are other colours which also have k candies. However, because we are only counting cases with $k \geq 11$, this approach poses no such difficulties.

We will work out each case separately and then add them to find the total number of ordered rolls which are unsandwichable.

Cases With 15 Candies of One Colour

It is easy to see that there are 6 rolls made up of a single colour, even without using the expression above.

Cases with 14 candies of one colour

$$\binom{15}{14} \cdot 6 \cdot 5^1 = 450$$

Cases with 13 candies of one colour

$$\binom{15}{13} \cdot 6 \cdot 5^2 = 15,750$$

Cases with 12 candies of one colour

$$\binom{15}{12} \cdot 6 \cdot 5^3 = 341,250$$

Cases with 11 candies of one colour

$$\binom{15}{11} \cdot 6 \cdot 5^4 = 5,118,750$$

Overall Probability

As we will show in the next section, every Smarties® roll with at most 10 candies of any one colour is sandwichable. Therefore, the total number of unsandwichable rolls can be calculated by summing the above. The result is 6 + 450 + 15, 750 + 341, 250 + 5, 118, 750 = 5,476, 206. The total number of all possible ordered rolls is 6^{15} =470,184,984,576.

Dividing the unsandwichable rolls by the total possible rolls, we find that the probability of finding an unsandwichable roll is about 0.0000116, or .00116%. The Smarties® company sells around 2 billion rolls a year*, so they should produce about 23,000 unsandwichable rolls every year.

Sandwichability Conditions

Instead of discussing only the case of a roll of Smarties® with 15 candies and 6 colours, in this section we characterize sandwichability for rolls of various lengths and numbers of colours.

Theorem 1 (General Sandwichability Theorem)

Given a roll of Smarties® of length $3n$ for some $n \in \mathbb{N}$, the roll is sandwichable if and only if the following two conditions are fulfilled:

1. There are no more than $2n$ Smarties® of any given colour.

2. There are no more than n colours containing an odd number of Smarties®.

Proof

Showing that the roll is unsandwichable if the two conditions above are not met is straightforward. Imagine beginning with zero of each colour and building the Smarties® roll sandwich by sandwich. Each of the n sandwiches can use at most two of a single colour, so we can have at most $2n$ of any given colour. Adding a bun to a colour cannot change its parity. Therefore, a sandwich can only change the parity of one colour with its filling, and since we started with all even colours, we will end up with at most n odd colours. Thus, any roll that does not meet the two conditions above will be unsandwichable.

Next, we will prove that these two conditions are sufficient to guarantee sandwichability. First, we need to show that we can produce n buns. We take away one candy from each odd colour. The conditions above guarantee that we will be left with at least $2n$ candies comprised of even colours, so these candies can be formed into n buns.

Next, we will take n arbitrary buns and assign fillings to them arbitrarily. If all of the sandwiches that we have made are legal, we are done. Suppose, then, that there is some illegal sandwich whose filling and bun are the same colour, say, orange. Not all of the buns are orange, because if they were and the filling of the illegal sandwich was also orange, there would be at least $2n + 1$ orange candies. This contradicts our assumptions.

So let us take some sandwich whose bun is some colour besides orange, say, red. There are two cases. In Case 1, the filling of the red bun is some colour besides orange, say, green. Then we will put the green filling in the orange bun and the orange filling in the red bun. In Case 2, the filling of the red bun is orange. Then we have four orange candies and two red candies in the two sandwiches, so we will make these into two sandwiches, each with an orange bun and a red filling.

* *https://money.cnn.com/2015/10/04/investing/ smarties-candy-company-millennial-women/*

Using the procedure above, we can always reduce the number of illegal sandwiches by one. We will do this for each illegal sandwich until all of the sandwiches are legal and the roll is sandwiched.

Thus, any roll of length $3n$ that meets these two conditions is sandwichable.

Corollary 1
If and only if a standard roll of 15 Smarties® has no more than 10 candies of a single colour, it is sandwichable.

Proof:
Recall that a standard roll of Smarties® can have up to six colours. There cannot be six odd colours, for then there would be an even number of Smarties® in the roll. The very thing is absurd because 15 is not even. Therefore, there are at most five odd colours. The condition that there are no more than 10 candies of a single colour is the other condition of the General Sandwichability Theorem. Therefore, a standard Smarties® roll is unsandwichable if and only if there are more than 10 candies of a single color. $O\pi\epsilon\rho\ E\delta\epsilon\iota\ \Delta\epsilon\iota\xi\alpha\iota$

Trivia

- Smarties® are known as Rockets in Canada because Canada already had a different candy by the same name
- Classic Smarties® flavours:

  Red (listed as pink on the Smarties® website) = Cherry
  Orange = Orange
  Yellow = Pineapple
  Green = Strawberry
  Purple = Grape
  White = Orange−Cream

- Smarties® were invented in 1949 when pellet machines from the war were purchased by the Ce De Candy company and used to make candy
- Smarties® webpage: https://www.smarties.com/
- Video of how Smarties® are made inside a Smarties® factory:
  https://www.youtube.com/watch?v= PhDux1hdLOY

Photo: @DiamondRehabThailand

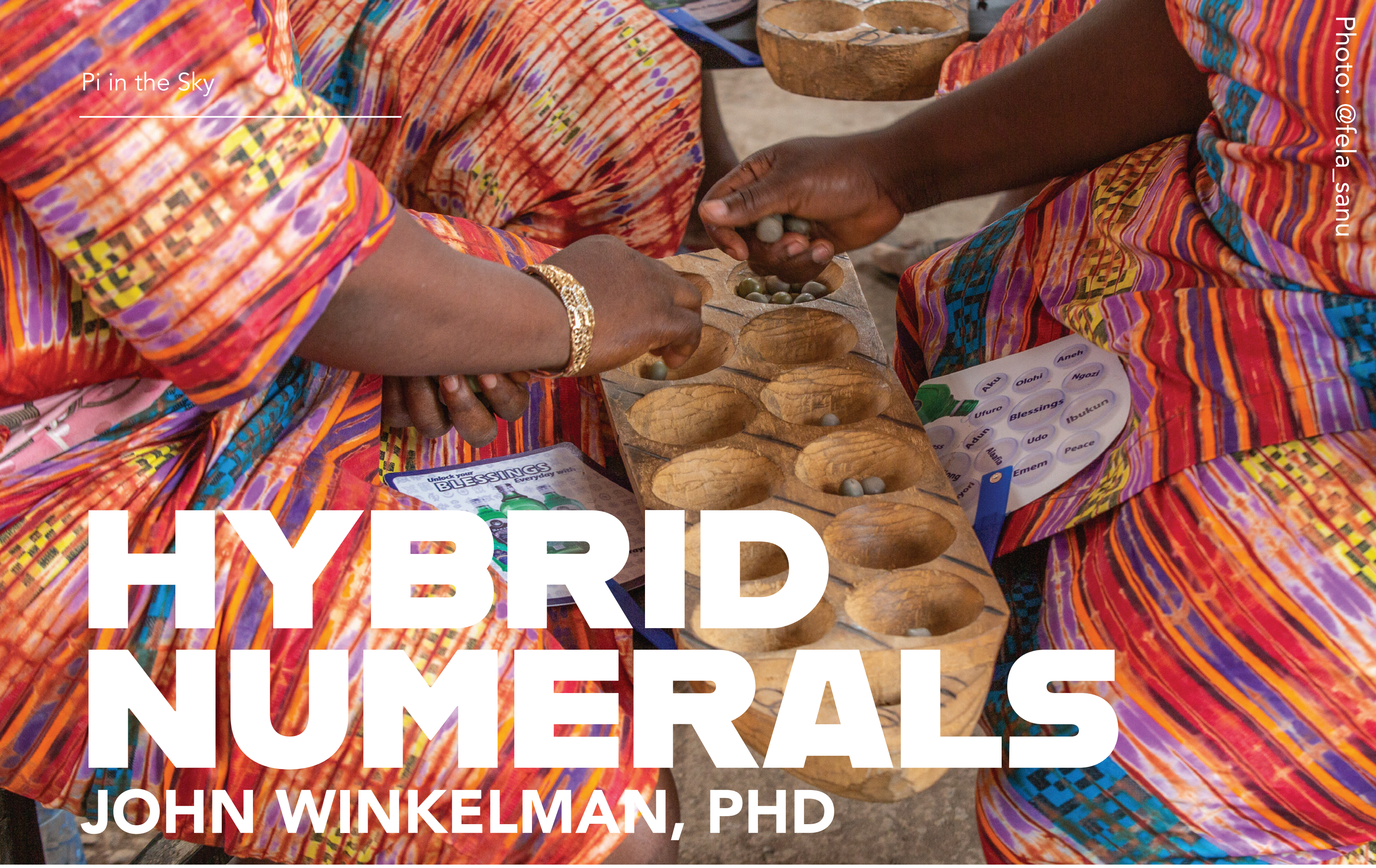

Photo: @fela_sanu

HYBRID NUMERALS

JOHN WINKELMAN, PHD

There are many ways to express numbers. In our usual base 10 place system, integers are expressed in the form a+ bx10 + cx100 etc., e.g. 1532 = 2 + 3x10 + 5x100 + 1x1000. a, b, c etc. have to be integers drawn from the set N={0,1,2,3,4,5,6,7,8,9}. Every positive integer can be expressed uniquely in this way, provided we exclude leading 0's, e.g. we would not write 01532 because the 0 would add no information.

One variation on our usual notation is to use a different base, for example 8 instead of 10. In base 8 1532 would become 2774, meaning 4+7x8 + 7x64 + 2x512. It looks different, but is still the same number. The base 10 system is not any better or worse than a base 8 system. They both work fine. We just happen to use 10.

In this paper I will look at another variation of the base 10 place system, inspired by the use of subtraction in the Nigerian language Yoruba. A Yoruba speaker will express 15 as 20 – 5, 16 as 20 – 4, and so on, up to 19 as 20 – 1. Yoruba does this quite extensively for numbers larger than 15. Roman numerals do something similar by expressing 4 as IV = 5 – 1, 9 as IX, i.e. 10 take 1, 90 as XC, 100 – 10. In English we express the clock time 12:45 as "a quarter to one", i.e. 1:00 – :15.

We could express the Yoruba version of 15 as 25', where 5' means negative 5, i.e. as 2x10 + 5'x1. We could regard the Yoruba language as using the set Y= {0,1,2,3,4,5',4'.3',2',1'} instead of N above. (This oversimplifies Yoruba considerably, since it uses a base 20 system and doesn't use subtraction for numbers less than 15). We can express any number using Y. For example the number 93816 becomes 11'42'24' in Y. We can check that by separating the positive and negative digits : 11'42'24'= 104020 – 10204 = 93816.

Y is not the only possible mixed set of 10 positive and negative digits. We can ask what are the necessary and sufficient conditions for a mixed set S of 10 integers to work, i.e. to give a unique representation for any positive integer. For example, could we use X={0,1,2,3,4,5,5',7,8,1'}? No, because 5 has two representations, as 5 and 15', and there is no way to represent 6. We can also ask further questions about these "hybrid" numerals: how do negative numbers work, how do we perform ordinary arithmetic, how do we compare numbers, and how do we represent fractions as decimals.

We will call a set of 10 digits "adequate" if and only if it contains 0,1, and exactly one of each of 2,8'; 3,7'; 4,6'; 5,5'; 6,4'; 7,3'; 8,2'; and 9,1', in other words one each of n or (n − 10) for n=2 to 9. N and Y were adequate. X={0,1,2,3,4,5,5',7,8,9} is not because it contains both 5 and 5', and also lacks both of 6 and 4'.

Theorem 1

For a set S of 10 integers to represent all positive integers uniquely, it is necessary and sufficient for S to be adequate.

In the proof we will assume that all positive integers can be represented uniquely by N.

We will prove theorem 1 by induction, a very useful proof method known by all mathematicians. For those not familiar with it, it goes like this.

We want to prove a proposition about positive integers. Call it P(n). We first prove that P(1) is true. We then show that if P(k) is true, P(k+1) must also be true. This means that P(1) implies P(2) implies P(3) and so on. P(n) has been then shown to be true for all n.

We will use an equivalent version in which we prove P(1), and then show that if P(1), P(2)…,P(k) are assumed, then P(k+1) must be true. Again, P(1) will imply P(2); together they imply P(3), and so on.

In our proof P(k) will mean that theorem 1 is true for all integers less than 10^k, including 0.

P(1): For S to represent the integers 0 to 9 uniquely, it is necessary and sufficient for S to be adequate.

Necessary. If S is not adequate, it cannot represent the integers 0 to 9. Suppose S lacks 0 or 1. If it lacks 0, it cannot represent 0, likewise if it lacks 1 it cannot represent 1 (e.g. 1 could be 19', but then S must contain 1). If S lacks any of the pairs n,(n − 10), then it cannot represent n. If it has both of them, its representation of n is not unique.

Sufficient. If S is adequate, it can represent all of 0 to 9 uniquely. It can represent 0 and 1 uniquely. Any other n from 2 to 9 must be one of n or 10 + (n − 10). For example 3 must be one of 3 or 17'. It can't be 1x' for another x' since 10+y=3 has only y=7' as a solution.

We next assume that P(1), P(2),...P(k) are true and show that this implies P(k+1). In other words, we assume any $n<10^k$ can be represented in an adequate S, and need to prove this is also true for all $n < 10^{(k+1)}$.

Let $n<10^{(k+1)}$. then n = 10A + B, where $A<10^k$ and B<10. (For example, 1328= 10 x 132 + 8). For convenience we will represent A as R(A) when we convert A from N to S. For example if S=Y, then R(3558) = 44'4'2'. If we write R(A) R(B) we mean the concatenation of R(A) then R(B). For example R(35584)= R(3558)R(4).

By the induction assumption we can represent A and B in S, so we can represent 10A and B in S. We will call these representations in S R(10A) and R(B). R(10A)= R(A)0. R(B) is of the form x or 1y', where x or y is one of N.

If B is in S then n is represented in S as R(A)R(B) (there is no carry).

If B is of the form 1y'=10 + y', n= 10A + 10 + y', where y' is in S. 10A + 10 =10 (A+1). By our induction hypothesis, we can represent (A+1) in S if A+1 <10^k, i.e. A<10^k-1.

If A<10^k-1, we can represent n in S as R(A+1)R(B).

If A= 10^k -1, then A+1=10^k. We can represent this A in S as well, as 1 followed by k 0's, since 0 and 1 must be in S. Therefore we can represent n as R(A+1)R(B).

Thus if S is adequate, it can represent any positive integer uniquely. If S is not adequate, it cannot even represent all integers <10 uniquely.

Converting n to a hybrid representation

The fact that any positive integer in N can be represented in a hybrid set S does not tell us how to do it. We can start with n represented in N, i.e. with the usual positive digits. We can then convert this representation from right to left, to end up with a representation in S.

For example, we wish to express n= 152928 using Y. This is n=(15292) x 10 + 8. 8 is 12' in Y, i.e. 10 + 2'. n becomes (15292+1) x10+2'.We use the same method on (15293) = 1529 x 10 + 3. The end result is 25'31'32'. To convert in the other direction, from S to N, we simply divide the S representation into the difference of two positive integers in N and subtract. Thus 25'31'32'= (203030 – 50102)= 152928 in N.

The number of adequate sets

A little reflection shows that there are 2^8 = 256 adequate sets. Only one of them, N, consists solely of positive integers, the other 255 have at least one negative integer. N lies at one extreme, while the set Z={0,1,2',3',4',5',6',7',8',1'} lies at the other.

Z would represent the digits 2 to 9 as 18',17',…,11'., and the number 1492 as 18'5'08'.

Negative integers

Can an adequate S represent negative integers without an initial minus sign? It turns out it can, but only if S has 1'. This excludes half of the adequate sets S, namely those which have a 9.

Theorem 2

S can represent all negative integers uniquely, without a minus sign, if and only if S is adequate and contains 1'.

We borrow from the method of proof used in Theorem 1. First, we show that an adequate S can represent the negative integers -1,-2,..-9 if and only if S contains 1'. We then show that S can represent all negative integers, using an inductive proof.

First, it is necessary for S to contain 1', otherwise it cannot represent -1.

To do so 1' would have to be 2'x, but no x will work. We show it is sufficient. Consider – n, where n is 2 to 9. If n' is in S, we are done. If n' is not in S, then d= 10 – n is in S, and 1'd represents –n. For example, if 6' is not in S, then 4 is, and – 6 = 1′4.

Next we show that being able to represent -1 to -9 is necessary and sufficient to represent all negative integers. The method is the same as in the proof of P1, so we omit it in the interest of brevity.

For example, we convert -3816 to hybrid Y, which includes 1′.
-3816= 3′8′1′6′ . We then convert 6′=1′4 and 8′=1′2 in Y.
3′8′1′6′= 3′8′2′4=4′22′4.

Arithmetic operations

Can we perform the ordinary arithmetic operations using hybrid representations? Yes, and we use the same algorithms as for conducting the operations in N. It will be cumbersome using hybrid numerals since we are not used to them. I will illustrate with examples using Y as the adequate set.

Addition. To add m + n, we represent m as 10A+B and n as 10C+D, i.e. we separate the final digits B,C and the leading groups of digits A,C. Schematically, we add 10A+B + 10C+D as 10(A+C)+ (B+D), which is what we do in ordinary arithmetic. B+D may give a carry of 1 or 1′ or no carry. We next evaluate 10(A+C+x) where x=0,1,or 1′, using the same procedure. At each step we have fewer digits, so the procedure will terminate.

Example: X= 15′2′01′5′ + 34′12′3′
A=15′2′01′, B= 5′, C= 34′12′ , D=3′
X= 10x (A+C)+ B+D
A+C= (15′2′01′+34′12′) =13′413′ (details omitted)
B+D = 5′+3′ =1′2 so we carry 1′
(A+C+1′) = 13′414′
Thus X= 13′414′2
(47985+26077= 74062)

Summary:

The use of subtraction in number representation in Yoruba and in Roman numerals suggests the possibility of using some negative integers in a base 10 place system. We found that this works only if the representing set S is “adequate”. We also found that we can represent negative numbers without using a minus sign if S is adequate and contains 1′ (negative one). We briefly showed how to translate decimals in N to a hybrid set.

The addition and multiplication tables can be reduced by using properties of negative numbers; e.g. in Y we only need rows and columns for 0,1,2,3,4,5′. Others, such as 13′x15′ can be deduced using arithmetic operations. However, this savings is offset by the greater computational effort if there are many negative digits. Hybrid systems which include only 1′ as the negative digits integrate negative integers into the whole set of integers. A system including only 1′, in place of 9, would simplify numbers such as 999, which would become 1001′. The 9’s tables are simplified as well: 5 x 9 = 5 x 11′ = 55′, 9 x 9= 11′ x 11′=12′1. Symmetries between 9 and 11 become evident.

Hybrid systems might have some use in disguising numbers by adding another level of encryption. Having said this the main point of interest of hybrid representations is not their practical use, which remains to be determined, but their existence as a natural extension of N. As in nature, most innovations are inferior to the original, but some survive and proliferate, perhaps in a niche.

2024 MATH QUICKIES

1. A jacket was originally priced \$200. The price was reduced 20% three times and increased 10% two times in some order. To the nearest cent, find the final price of the jacket.

2. In a right prism, the base is a right-angled triangle having one leg equal to the height of the prism. The sum of the lengths of the other leg and the hypotenuse is 10. Find the maximum possible volume of the prism.

3. The reflection of the point *A* - *(-1, 1)* across line $2x + y = 1$ is $A' = (a,b)$. Find a and b.

4. If $a_0, a_1, \dots, a_n, \dots$ denote the sequence of real numbers such that $a_1 = 3$ an $a_{n+1} = \frac{a_n}{1+a_n}$. Find the value of a_{199}.

5. Find all the points *(x,y)* with x, y integers, which are inside the circle $x^2 + y^2 = 1000$ and such that $|3x+1| + |x-1| = y$.

6. Find the number of integers m, such that $0 \leq m \leq 1000$ and the solution of the equation $x^2 + 3x - m = 0$ are integers.

7. Find the 101^{st} positive integer that cannot be written as a difference of squares.

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PIMS is sponsoring a prize of \$100 CAD to the first high school student (from within the PIMS operating region: Alberta; British Columbia; Manitoba; Saskatchewan; Oregon; Washington) who submits the largest number of correct answers before June 1, 2025. Submit your answers to: s.demirbas@math.ubc.ca.

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